Calculation of Contact Forces

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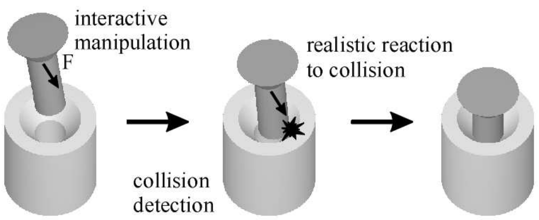
Abstract

Detecting collisions and calculating physically correct collision responses play an important role when simulating the dynamics of colliding rigid bodies. VR-applications such as virtual assembly planning and ergonomy studies can especially profit from advances in these directions, because they enable an interactive and intuitive manipulation of objects in virtual environments. This paper presents new algorithms for the calculation of contact forces in multi-body systems with unilateral contacts.

1 Introduction

In order to interactively simulate an assembly of mechanical parts in a VR-environment, the simulation software must be able to detect collisions and calculate reaction forces efficiently. The following figure demonstrates the principle:



The manual insertion of a bolt into a hole is a difficult task in a virtual environment, if the motion simply stops as soon as a collision occurs. In reality the bolt automatically slides into the right direction when it comes into contact with the conical boundary of the hole. It is desirable to simulate this effect in order to perform virtual fitting operations in a more intuitive manner.
Our paper describes two different mathematical approaches to determine the reaction forces for colliding rigid bodies.

2 Calculation of contact forces

Physically accurate models for the determination of contact forces use a complementarity formulation, which reflects the unilateral nature of the contacts [2, 5, 1, 3, 4]. These methods are well suited for situations with multiple contact points and can be extended to handle friction.

Let us consider a multi-body system consisting of n rigid bodies in mutual contact at K contact points. Suppose that body B_{i_k} touches body B_{j_k} at the k-th contact point $\boldsymbol{p}_k$. The interpenetration of these bodies at $\boldsymbol{p}_k$ is prevented by a pair of opposite directed contact forces $\pm\boldsymbol{F}_k = \pm f_k \boldsymbol{n}_k$. In the absence of friction $\boldsymbol{F}_k$ acts in direction of the normal vector $\boldsymbol{n}_k$ of the contact plane, which is tangential to the surface of both objects at the contact point. Let the vector $\boldsymbol{r}_{kl} = \boldsymbol{p}_k - \boldsymbol{c}_l$ point from the center of mass of object B_l to the k-th contact point. The position and orientation of object B_l are described by the vector $\boldsymbol{c}_l$ and a quaternion $\boldsymbol{q}_l$. $\boldsymbol{v}_l$ and $\boldsymbol{\omega}_l$ specify the linear and angular velocity of body B_l and m_l and $\mathbf{I}_l$ denote its mass and its inertia matrix.

Our objective is to determine the constraint-forces $\boldsymbol{F}_k$ for $k = 1, \ldots, K$. To give up the component-wise description the following vectors and matrices are quite useful: the generalized velocity vector $\boldsymbol{u} = [\boldsymbol{v}_1, \boldsymbol{\omega}_1, \ldots, \boldsymbol{v}_n, \boldsymbol{\omega}_n]^T \in I\!R^{6n}$, the generalized position vector $\boldsymbol{s} = [\boldsymbol{c}_1, \boldsymbol{q}_1, \ldots, \boldsymbol{c}_n, \boldsymbol{q}_n]^T \in I\!R^{7n}$, the vector of the magnitudes of the contact forces $\boldsymbol{f} = [f_1, f_2, \ldots, f_K]^T \in I\!R^K$, the vector of external forces $\boldsymbol{f}_{ext} = [m_1\boldsymbol{g}, -\boldsymbol{\omega}_1 \times \mathbf{I}_1\boldsymbol{\omega}_1, \ldots, m_n\boldsymbol{g}, -\boldsymbol{\omega}_n \times \mathbf{I}_n\boldsymbol{\omega}_n]^T \in I\!R^{6n}$, the matrix $\mathbf{S} = \operatorname{diag}(\mathbf{E}, \mathbf{Q}_1, \ldots, \mathbf{E}, \mathbf{Q}_n) \in I\!R^{7n \times 6n}$, where $\mathbf{Q}_l \in I\!R^{4\times 3}$ imitates the quaternion product $\frac{1}{2}\boldsymbol{\omega}_l \boldsymbol{q}_l = \mathbf{Q}_l \boldsymbol{\omega}_l$, the generalized mass matrix $\mathbf{M} = \operatorname{diag}(m_1\mathbf{E}, \mathbf{I}_1, \ldots, m_n\mathbf{E}, \mathbf{I}_n) \in I\!R^{6n\times 6n}$, the matrix $\mathbf{N} = \operatorname{diag}(\boldsymbol{n}_1, \ldots, \boldsymbol{n}_K) \in I\!R^{3K\times K}$ of contact normals and the matrix of contact conditions $\mathbf{J} \in I\!R^{6n\times 3K}$. The transposed matrix $\mathbf{J}^T$ has the following structure indicated by its k-th row:

$$\begin{array}{ccccccc} & {\scriptstyle 2i_k-1} & {\scriptstyle 2i_k} & & {\scriptstyle 2j_k-1} & {\scriptstyle 2j_k} & \\ & \downarrow & \downarrow & & \downarrow & \downarrow & \\ \end{array}$$
$$\begin{bmatrix} \mathbf{0}\ldots\mathbf{0} & -\mathbf{E} & \mathbf{r}^{\times}_{ki_k} & \mathbf{0}\ldots\mathbf{0} & \mathbf{E} & -\mathbf{r}^{\times}_{kj_k} & \mathbf{0}\ldots\mathbf{0} \end{bmatrix}.$$

$\mathbf{r}^{\times} \in I\!R^{3\times 3}$ is the skew-symmetric matrix with $\mathbf{r}^{\times}\boldsymbol{a} = \boldsymbol{r} \times \boldsymbol{a}$ for $\boldsymbol{a} \in I\!R^3$.

Using this notation, the Newton-Euler equations of motion can be formulated in their continuous and discretized (Euler-scheme) version:

$$\begin{aligned} \dot{\boldsymbol{s}} &= \mathbf{S}\boldsymbol{u} \\ \dot{\boldsymbol{u}} &= \mathbf{M}^{-1}(\mathbf{J}\mathbf{N}\boldsymbol{f} + \boldsymbol{f}_{ext}) \end{aligned}$$

$$\Rightarrow \boldsymbol{s}^{t+\Delta t} = \boldsymbol{s}^t + \Delta t \mathbf{S}\boldsymbol{u}^{t+\Delta t} \quad (1)$$

$$\boldsymbol{u}^{t+\Delta t} = \boldsymbol{u}^t + \Delta t \mathbf{M}^{-1}(\mathbf{J}\mathbf{N}\boldsymbol{f} + \boldsymbol{f}_{ext}) \quad (2)$$

Now we define the function $\boldsymbol{\delta} : I\!R^{7n} \to I\!R^K$ that computes the contact distances $\boldsymbol{\delta}(\boldsymbol{s}) = [\delta_1, \ldots, \delta_K]^T$ for the generalized position vector $\boldsymbol{s}$. Then δ_k is the distance between the parts of the objects involved in the k-th (potential) contact. Note, that $\boldsymbol{\delta}$ is non-linear.

Finally, we need the function $\boldsymbol{\sigma} : I\!R^K \to I\!R^{7n}$ taking the magnitudes of the contact forces as arguments and computing the configuration $\boldsymbol{s}^{t+\Delta t}$. We obtain $\boldsymbol{\sigma}$ by inserting equation (2) into equation (1):

$$\boldsymbol{\sigma}(\boldsymbol{f}) = \boldsymbol{s}^t + \Delta t \mathbf{S}(\boldsymbol{u}^t + \Delta t \mathbf{M}^{-1}(\mathbf{J}\mathbf{N}\boldsymbol{f} + \boldsymbol{f}_{ext}))$$

2.1 Reduction to a system of equations

If we assume that all contacts are bilateral, the condition

$$\boldsymbol{\delta}\left(\boldsymbol{\sigma}(\boldsymbol{f})\right) = \mathbf{0} \tag{3}$$

with $\mathbf{0} = [0, \ldots, 0]^T \in I\!R^K$ must hold. This is a non-linear equation system with the contact forces as unknown quantities. E.g. it can be solved by the Newton-Raphson method, which requires the successive solution of a K dimensional linear system of equations. Since the contacts are not really bilateral, we release the k-th contact if f_k becomes negative.

Now suppose that all contacts are unilateral. Then a disadvantage of the above method is that we allow negative contact forces for one simulation step, which cause the objects to stick together for a short time. We can avoid this by replacing condition (3) by the complementarity condition

$$\boldsymbol{\delta}\left(\boldsymbol{\sigma}(\boldsymbol{f})\right) \geq \mathbf{0} \quad \text{compl. to} \quad \boldsymbol{f} \geq \mathbf{0}. \tag{4}$$

Note that '$\boldsymbol{a}$ compl. to $\boldsymbol{b}$' is equivalent to $\boldsymbol{a}^T\boldsymbol{b} = 0$ for $\boldsymbol{a}, \boldsymbol{b} \in I\!R^d$. Condition (4) means that we do not allow negative distances (i.e. interpenetration) or negative (i.e. attractive) contact forces and that at each contact point the distance or the force must be equal to zero. In order to solve this non-linear complementarity problem (NCP) we use the same technique as in [3] and consider the so-called Fischer function $\varphi : I\!R^2 \to I\!R$ which is defined by $\varphi(a,b) = \sqrt{a^2+b^2} - a - b$. It is obvious that the following holds for each $a, b \in I\!R$: $\varphi(a,b) = 0$ iff $a \geq 0$, $b \geq 0$ and $ab = 0$. We define the function $\boldsymbol{\Phi} : I\!R^{2K} \to I\!R^K$ by

$$\boldsymbol{\Phi}(\boldsymbol{f}) = [\varphi(f_1, \delta_1(\sigma(\boldsymbol{f}))), \ldots, \varphi(f_K, \delta_K(\sigma(\boldsymbol{f})))]^T$$

Now we can transform the NCP into the non-linear equation system

$$\boldsymbol{\Phi}(\boldsymbol{f}) = \mathbf{0}, \tag{5}$$

which can again be solved by the Newton-Raphson method. In contrast to the classical method [2] this approach uses the contact distances instead of the contact accelerations as variables complementary to the contact forces. In this way the integration of the motion equations can be performed in a stable way with respect to the geometric constraints. The classical method however has to face the problem that the deviations from the exact constraints accumulate during the integration process.

2.2 Solution by a fixpoint iteration

Now we present an alternative method for the calculation of the contact forces which also guarantees the compliance with the contact conditions. It is derived from [5] and [4] which solve a sequence of linear complementarity problems using a fixpoint iteration.

Let us consider the normal component of the relative contact velocity of the k-th contact point which is given by:

$$\boldsymbol{n}_k^T(\boldsymbol{v}_{j_k} + \boldsymbol{\omega}_{j_k} \times \boldsymbol{r}_{kj_k}) - \boldsymbol{n}_k^T(\boldsymbol{v}_{i_k} + \boldsymbol{\omega}_{i_k} \times \boldsymbol{r}_{ki_k})$$

Formulated for all contacts, this is equal to $\mathbf{N}^T\mathbf{J}^T\boldsymbol{u}$. Now for the velocities $\boldsymbol{u}^{t+\Delta t}$ the complementarity condition

$$\mathbf{N}^T\mathbf{J}^T\boldsymbol{u}^{t+\Delta t} \geq \frac{\boldsymbol{\nu}}{\Delta t} \quad \text{compl. to} \quad \boldsymbol{f} \geq \mathbf{0} \tag{6}$$

must hold with $\boldsymbol{\nu} = [\nu_1, \ldots, \nu_K]^T \in I\!R^K$. We insert equation (2) into equation (6) and obtain:

$$\begin{aligned} & \mathbf{N}^T\mathbf{J}^T\mathbf{M}^{-1}\mathbf{J}\mathbf{N}\Delta t\boldsymbol{f} \\ + \quad & \mathbf{N}^T\mathbf{J}^T(\boldsymbol{u}^t + \Delta t\mathbf{M}^{-1}\boldsymbol{f}_{ext}) - \frac{\boldsymbol{\nu}}{\Delta t} \geq \mathbf{0}\,. \end{aligned}$$

We choose $\boldsymbol{\nu} = -\boldsymbol{\delta}(\boldsymbol{s}^{t+\Delta t}) + \Delta t\mathbf{N}^T\mathbf{J}^T\boldsymbol{u}^{t+\Delta t}$. This is a linear complementarity problem (LCP) of the form

$$\mathbf{A}\boldsymbol{f} + \boldsymbol{b} \geq \mathbf{0} \quad \text{compl. to} \quad \boldsymbol{f} \geq \mathbf{0} \tag{7}$$

with $\mathbf{A} = \mathbf{N}^T\mathbf{J}^T\mathbf{M}^{-1}\mathbf{J}\mathbf{N} \in I\!R^{K\times K}$ and $\boldsymbol{b} \in I\!R^K$. It can be solved with the classical Lemke-algorithm. $\mathbf{A}$ is symmetric and positive semidefinite, because the generalized mass matrix $\mathbf{M}$ has these properties.
The linearization makes the solution of (7) too inaccurate to be used directly. For reasons of numerical stability it is important to fulfil the geometric constraint $\boldsymbol{\delta}(\boldsymbol{s}^{t+\Delta t}) \geq \mathbf{0}$ exactly. This is guaranteed by the choice of $\boldsymbol{\nu}$. Therefore we perform the following fixpoint iteration.

$\boldsymbol{u}' \leftarrow \boldsymbol{u}^t + \Delta t\mathbf{M}^{-1}\boldsymbol{f}_{ext}$
$\boldsymbol{s}' \leftarrow \boldsymbol{s}^t + \Delta t\mathbf{S}'\boldsymbol{u}'$
$\boldsymbol{f} \leftarrow \mathbf{0}$
repeat
  $\boldsymbol{f}' \leftarrow \boldsymbol{f}$
  $\boldsymbol{f} \leftarrow LCP(\boldsymbol{s}', \boldsymbol{u}')$
  $\boldsymbol{u}' \leftarrow \boldsymbol{u}^t + \Delta t\mathbf{M}^{-1}(\mathbf{J}\mathbf{N}\boldsymbol{f} + \boldsymbol{f}_{ext})$
  $\boldsymbol{s}' \leftarrow \boldsymbol{s}^t + \Delta t\mathbf{S}'\boldsymbol{u}'$
until $|\boldsymbol{f}' - \boldsymbol{f}| < \varepsilon$
$\boldsymbol{s}^{t+\Delta t} \leftarrow \boldsymbol{s}'$

$LCP(.,.)$ describes the Lemke-algorithm and the crucial part of its input.

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